Limits of functions of 2 variables are denoted

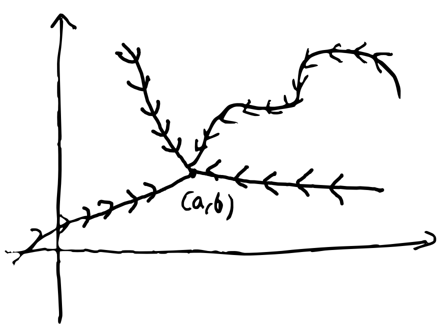
$$\boxed{\lim_{(x,y)\to(a,b)} f(x,y)}$$

<u>Definition</u> We say $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if,

as (x,y) gets closer to (a,b),

$f(x,y)$ gets closer to L.

Here, (x,y) getting closer to (a,b) means you take a 2D path that leads to (a,b) and let (x,y) move along that path. The limit exists if the limit along a path to (a,b) is the same <u>regardless of the choice of the path.</u>



Unlike the single-variable limit, there are extremely many different choices of paths. <u>You may never argue that a 2-variable limit exists by checking on all possible paths.</u>

You may still use paths to argue that the <u>limit does not exist</u> (<u>DNE</u> in short). The limit does not exist if either

- there is <u>one path</u> along which <u>the limit does not exist</u>,

or

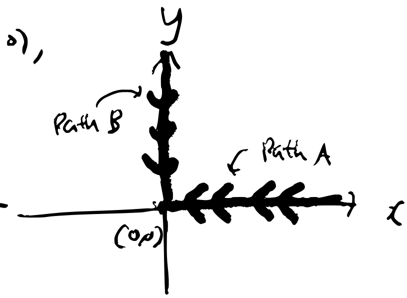
- there are <u>two paths</u> whose <u>limits are different</u>.

<u>Example</u> Show $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does not exist.

Consider the two paths to $(0,0)$,

<u>Path A: Along the positive x-axis.</u>

<u>Path B: Along the positive y-axis.</u>



Path A means $y=0$, $x\to 0$.

$$\boxed{\lim_{\text{Path A}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-0^2}{x^2+0^2} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1}$$

Path B means $x=0$, $y\to 0$.

$$\boxed{\lim_{\text{Path B}} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} \frac{0^2-y^2}{0^2+y^2} = \lim_{y\to 0} -\frac{y^2}{y^2} = \lim_{y\to 0} -1 = -1.}$$

The two limits are different, so the limit DNE.

Example Show $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ DNE.

Well, let's again try

Path A: Along the positive x-axis.

Path B: Along the positive y-axis.

Path A means $y=0$, $x\to 0$.

$$\boxed{\lim_{\text{Path A}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x\cdot 0}{x^2+0^2} = \lim_{x\to 0} 0 = 0.}$$

Path B means $x=0$, $y\to 0$.

$$\boxed{\lim_{\text{Path B}} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{0\cdot y}{0^2+y^2} = \lim_{y\to 0} 0 = 0.}$$

So, the both paths have the same limit..

To show that the limit DNE, we need a different path.

Path C: Along $y=x$.

Path C means $y=x$, $x\to 0$.

$$\boxed{\lim_{\text{Path C}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x\cdot x}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.}$$

We get a different limit, so the limit DNE.

<u>Example</u> Show $\lim_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$ DNE.

Let's try the paths we've used.

Path A: $y=0$, $x\to 0$

$$\boxed{\lim_{\text{Path A}} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2\cdot 0}{x^4+0^2} = \lim_{x\to 0} 0 = 0}$$

Path B: $x=0$, $y\to 0$

$$\boxed{\lim_{\text{Path B}} \frac{x^2y}{x^4+y^2} = \lim_{y\to 0} \frac{0^2\cdot y}{0^4+y^2} = \lim_{y\to 0} 0 = 0}.$$

Path C: $y=x$, $x\to 0$

$$\lim_{\text{Path C}} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2\cdot x}{x^4+x^2} = \lim_{x\to 0} \frac{x^3}{x^4+x^2} = \lim_{x\to 0} \frac{x}{x^2+1}$$

$$= \frac{0}{0^2+1} = 0.$$

They all lead to the same limit...

In fact, for this you need to use a "curved path"!

<u>Path D: Along $y=x^2$.</u>

Path D means $y=x^2$, $x\to 0$, so

$$\boxed{\lim_{\text{Path D}} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2\cdot x^2}{x^4+(x^2)^2} = \lim_{x\to 0} \frac{x^4}{2x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}}$$

We get a different limit, so the limit DNE.

As there are unimaginably many paths in 2D, it is impossible to show that the 2-variable limit exists by checking on all paths. To show that the limit exists, you rather have to use a different method.

Method 0. Plugging in.

The two-variable limit $\lim_{(x,y)\to(a,b)} f(x,y)$ exists if you can plug $(x,y)=(a,b)$ in the expression of $f(x,y)$ and make sense of it; in other words, when (a,b) is in the domain of $f(x,y)$.

If so, the limit $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Example $\lim_{(x,y)\to(0,0)} x^2+y^2 = 0^2+0^2=0$.

Plugging in is inconclusive if you obtain a bogus expression such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$. (Note that there is no 2-variable L'Hopital!)

Method 1 Continuity argument.

Sometimes one can convert $\lim_{(x,y)\to(a,b)} f(x,y)$ into a single-variable limit, where we have more tools.

If $f(x,y) = g_1(g_2(x,y))$ (every appearance of x, y is in terms of a single repeated expression $g_2(x,y)$).

and if $\boxed{\lim_{(x,y)\to(a,b)} g_2(x,y) = L}$ by some other reason,

then $\boxed{g_2 \to L \text{ as } (x,y)\to(a,b)}$. Thus

$$\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} g_1(g_2(x,y)) = \lim_{g_2\to L} g_1(g_2),}$$

which is a single variable limit!

Example Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = ?$$

Note that x^2+y^2 is repeated. So, for $f(x,y) = \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$,

$f(x,y) = g_1(g_2(x,y))$, where $g_1(g_2) = \frac{g_2}{\sqrt{g_2+1}-1}$,

$g_2(x,y) = x^2+y^2$. Note that

$$\lim_{(x,y)\to(0,0)} g_2(x,y) = \lim_{(x,y)\to(0,0)} x^2+y^2 = 0^2+0^2 = 0,$$ because you can just simply plug in the values.

So $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{g_2\to 0} \frac{g_2}{\sqrt{g_2+1}-1}$. As plugging $g_2=0$ gives $\frac{0}{0}$, we may use L'Hopital:

$$\lim_{g_2\to 0} \frac{g_2}{\sqrt{g_2+1}-1} \stackrel{\text{L'Hopital}}{=} \lim_{g_2\to 0} \frac{1}{\frac{1}{2\sqrt{g_2+1}}} = \lim_{g_2\to 0} 2\sqrt{g_2+1} = 2\sqrt{0+1} = 2.$$

So the limit is 2.

<u>Example</u> Find $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2y)}{x^2y} = ?$

For $f(x,y) = \frac{\sin(x^2y)}{x^2y}$, $f(x,y) = g_1(g_2(x,y))$, where $g_1(g_2) = \frac{\sin(g_2)}{g_2}$, $g_2(x,y) = x^2y$. Note $\lim_{(x,y)\to(0,0)} g_2(x,y) = \lim_{(x,y)\to(0,0)} x^2y = 0^2\cdot 0 = 0$, since you can simply plug in.

$$\text{So, } \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{g_2\to 0} \frac{\sin(g_2)}{g_2} \stackrel{\text{L'Hopital}}{=} \lim_{g_2\to 0} \frac{\cos(g_2)}{1} = \cos(0) = 1.$$

<u>Example</u> Find $\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2} = ?$

For $f(x,y) = \frac{e^{-x^2-y^2}-1}{x^2+y^2}$, $f(x,y) = g_1(g_2(x,y))$, where $g_1(g_2) = \frac{e^{-g_2}-1}{g_2}$, $g_2(x,y) = x^2+y^2$. Note that you can use plugging-in to get $\lim_{(x,y)\to(0,0)} x^2+y^2 = 0^2+0^2 = 0$. So, $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{g_2\to 0} \frac{e^{-g_2}-1}{g_2}$

$$\stackrel{\text{L'Hopital}}{=} \lim_{g_2\to 0} \frac{-e^{-g_2}}{1} = -e^{-0} = -1.$$

Method 2. Squeeze Theorem.

If $h_1(x,y) \le f(x,y) \le h_2(x,y)$, obviously

$$\boxed{\lim_{(x,y)\to(a,b)} h_1(x,y) \le \lim_{(x,y)\to(a,b)} f(x,y) \le \lim_{(x,y)\to(a,b)} h_2(x,y)}$$

Using this, we may compute $\lim_{(x,y)\to(a,b)} f(x,y)$ using other simpler limits. If we cleverly find $h_1(x,y) \le f(x,y) \le h_2(x,y)$ so that h_1, h_2 are simple, and such that the limits of h_1, h_2 are the same, then the limit of f is also the same.

When simplifying the expression, you may use:

- $-1 \le \sin \le 1$, $-1 \le \cos \le 1$
- If you remove a $+$ term from the denominator, the fraction gets larger.

<u>Example</u> Find $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^2} = ?$

Note $\frac{x^2y^2}{x^4+y^2} \ge 0$, and $\frac{x^2y^2}{x^4+y^2} \le \frac{x^2y^2}{y^2} = x^2$.

Remove x^4

So $0 \le \frac{x^2y^2}{x^4+y^2} \le x^2 \Rightarrow \lim_{(x,y)\to(0,0)} 0 \le \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^2} \le \lim_{(x,y)\to(0,0)} x^2$.

Since $\lim_{(x,y)\to(0,0)} 0 = 0$ (obvious), $\lim_{(x,y)\to(0,0)} x^2 = 0^2 = 0$ (plugging in),

$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^2} = 0$.

Example Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4}{x^2+y^2} = ?$

Clearly $\dfrac{x^4}{x^2+y^2} \geq 0$, and $\dfrac{x^4}{x^2+y^2} \leq \dfrac{x^4}{x^2} = x^2$

Remove y^2

$$\Rightarrow \lim_{(x,y)\to(0,0)} 0 \leq \lim_{(x,y)\to(0,0)} \frac{x^4}{x^2+y^2} \leq \lim_{(x,y)\to(0,0)} x^2$$

Since $\displaystyle\lim_{(x,y)\to(0,0)} 0 = 0$ (obvious), $\displaystyle\lim_{(x,y)\to(0,0)} x^2 = 0$ (plugging in)

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^4}{x^2+y^2} = 0.$$

You cannot remove all terms from the denominator.

You have to leave at least one term there.

Example Find $\displaystyle\lim_{(x,y)\to(0,0)} x^4 \sin(x+y) = ?$

Since $-1 \leq \sin(x+y) \leq 1$,

$$\Rightarrow -x^4 \leq x^4 \sin(x+y) \leq x^4$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} -x^4 \leq \lim_{(x,y)\to(0,0)} x^4 \sin(x+y) \leq \lim_{(x,y)\to(0,0)} x^4.$$

Since $\lim_{(x,y)\to(0,0)} -x^4 = 0$ (plugging in)

$\lim_{(x,y)\to(0,0)} x^4 = 0$ (plugging in)

$\Rightarrow \lim_{(x,y)\to(0,0)} x^4 \sin(x+y) = 0.$

<u>Example</u> (Hard) Find $\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2} = ?$

Obviously $0 \le \frac{x^4+y^4}{x^2+y^2}$. To find nice $h_2(x,y)$ such that $\frac{x^4+y^4}{x^2+y^2} \le h_2(x,y)$, you may split

$$\frac{x^4+y^4}{x^2+y^2} = \frac{x^4}{x^2+y^2} + \frac{y^4}{x^2+y^2}$$ and treat the two terms separately.

$$\frac{x^4}{x^2+y^2} \underbrace{\le}_{\text{Remove } y^2} \frac{x^4}{x^2} = x^2, \quad \frac{y^4}{x^2+y^2} \underbrace{\le}_{\text{Remove } x^2} \frac{y^4}{y^2} = y^2$$

$$\rightsquigarrow \frac{x^4+y^4}{x^2+y^2} = \frac{x^4}{x^2+y^2} + \frac{y^4}{x^2+y^2} \le x^2+y^2.$$

So $\lim_{(x,y)\to(0,0)} 0 \le \lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2} \le \lim_{(x,y)\to(0,0)} x^2+y^2.$

Since $\lim_{(x,y)\to(0,0)} 0 = 0$ (obvious), $\lim_{(x,y)\to(0,0)} x^2+y^2 = 0$ (plugging in),

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2} = 0.$$

You can apply the same methods to the limits of 3^+-variables functions.

<u>Example</u> Show that $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^4}$ does not exist.

Path 1: $y=z=0,\ x\to 0$

$$\boxed{\lim_{\text{Path 1}} \frac{xy+yz^2+xz^2}{x^2+y^2+z^4} = \lim_{x\to 0} \frac{x\cdot 0+0\cdot 0^2+x\cdot 0^2}{x^2+0^2+0^4} = \lim_{x\to 0} 0 = 0}$$

Path 2: $z=0,\ y=x,\ x\to 0$

$$\boxed{\lim_{\text{Path 2}} \frac{xy+yz^2+xz^2}{x^2+y^2+z^4} = \lim_{x\to 0} \frac{x\cdot x+x\cdot 0^2+x\cdot 0^2}{x^2+x^2+0^4} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}}$$

So, as the limits are different, the limit DNE.